QCD Instantons and 2D Surfaces

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Dedicated to the memory of
Prof. Dwight Nicholson, Prof. Bob Smith, Prof. Chris Goertz,
Dr. Anne Cleary and Dr. Linhua Shan,
and for the recovery of Ms. Miya Sonya Sioson.

ABSTRACT

Some time ago, Atiyah showed that there exists a natural identification between the k-instantons of a Yang-Mills theory with gauge group G and the holomorphic maps from CP_1 to ΩG. Since then, Nair and Mazur, have associated the Θ vacua structure in QCD with self-intersecting Riemann surfaces immersed in four dimensions. From here they concluded that these 2D surfaces correspond to the non-perturbative phase of QCD and carry the topological information of the Θ vacua. In this paper we would like to elaborate on this point by making use of Atiyah's identification. We will argue that an effective description of QCD may be more like a WZW model coupled to the induced metric of an immersion of a 2-D Riemann surface in R^4. We make some further comments on the relationship between the coadjoint orbits of the Kac-Moody group on G and instantons with axial symmetry and monopole charge.

One of the outstanding questions that still haunts theoretical physics today is that of the nature of the non-perturbative phase of QCD. Probes such as lattice calculations, $1/N_c$ expansions and topology, are beginning to reveal some of the salient features of the rich vacuum structure of QCD. In particular it is believed that a string theory may provide an effective description of QCD in this non-perturbative phase. Furthermore, it has been suggested that a string theory in four dimensions could enjoy a Θ-like term [1,2], and that this Θ term can be associated with the Θ vacua of QCD [3]. This term would correspond to the self-intersections of the image of a Riemann surface immersed in R^4. Thus one is able to capture an important feature of non-perturbative QCD in a string theory. In this note we would like to extend this list of qualitative features to include a correspondence between the moduli space of the maps associated with a 2D theory and the moduli space of the instanton sector of QCD. Indeed it has been known for some time [4] that not only is there a general correspondence between 2D structures and 4D Yang-Mills instantons, but that there also exists a diffeomorphism between the moduli spaces of certain maps from CP_1 into the loop group of G, and the moduli space of instantons on R^4. More specifically, Atiyah [4] has shown that the moduli space of 4D k-instantons is diffeomorphic to the moduli space of degree k, holomorphic maps from CP_1 into the loop group of G, viz ΩG, where G is the gauge group of the Yang-Mills theory. Thus one has a natural identification between 4D instantons and 2D surfaces.

We will exploit this identity to write down an action that may correspond to the instanton sector of QCD. First let us recall the overall features of the work of Atiyah. The main emphasis is on holomorphic maps, f, which carry CP_1 into ΩG, where ΩG in the loop group of the gauge group G of the Yang-Mills theory in question. The space ΩG consists of all maps from S^1 into G, $g : S^1 \rightarrow G$, such that $g(\theta = 0) = 1$. Thus the only constant map is the identity map. We will, of course, be primarily interested in the case when G is $SU(3)$ although the work can be extended to more general cases. Two important features of ΩG are that it is a Kähler manifold, and that it is infinitely dimensional. The relationship between the maps $f : CP_1 \rightarrow \Omega G$ and the Yang-Mills instantons arise from the fact that

a) By twistor methods, 4D instantons are described by holomorphic maps, $f : CP_1 \rightarrow \Omega G$ and

b) by using the results of Donaldson [5], one can show that all such holomorphic maps arise from these twistor methods. (Although the argument is proved only for classical groups, this is sufficient for our purposes.)

From the two statements above , Atiyah is able to prove the following theorem;

Theorem[Atiyah]: The moduli space of Yang-Mills k-instantons (k an integer) over R^4 with classical group G, modulo based gauge transformations and the moduli space of *all* based holomorphic maps from $CP_1 \rightarrow \Omega G$ of degree k, are diffeomorphic to each other. We will denote by $\mathcal{M}_{4D}$ and $\mathcal{M}_{2D}$ for the respective moduli spaces.

We note that the Yang-Mills moduli space mentioned in the above theorem is that which includes moduli associated with constant gauge transformations. To clarify this point, consider $G = SU(2)$. By acting on known solutions of the equations of motion with the symmetries of the Yang-Mills action one can find new solutions that may not be gauge related. Indeed by using the full conformal group (Poincarè transformations, and the conformal transformations) one can show that there are four parameters coming from the translations, one coming from the scale transformations, and three coming from the action of the constant gauge transformations. Thus there is an $8k$ dimensional moduli space for the k-instanton solutions. If we ignore the constant gauge transformations, this is tantamount to using all the gauge transformations (including the constant ones) in the equivalence relations. This effectively reduces the dimension of the moduli space to $8k-3$ dimensions [6]. This moduli space is denoted by $\mathcal{M}_{4D}/G$ and in general may not be a manifold due to a finite set of points that are related to the reducible connections.

Now let us focus on ΩG. It is well known that as a manifold ΩG is Kähler [7]. Furthermore, by using methods of Kirillov [8], one can show that it has a natural symplectic structure associated with a particular coadjoint orbit, namely the orbit associated with a pure centrally extended covector. In Ref. [9,10] we used this symplectic geometry to construct an action that on CP_1 is directly related to the natural Kähler metric on ΩG. By using the Kac-Moody algebra, which is the centrally extended algebra of ΩG, the action necessarily contains order $\hbar$ contributions. These order $\hbar$ contributions will be necessary for a realistic effective action of QCD.

The maps that we seek for the string action for QCD will be maps from $CP_1 \rightarrow \Omega G$. By using a general procedure [11], one is able to consider a two-dimensional manifold m where each point on this manifold corresponds to that element in ΩG which is used to transport a coadjoint vector, $\mathbf{b}$, to a new element $\mathbf{b_g}$ on the coadjoint orbit. The fields, $g(\sigma, \lambda, \tau)$, provide a map from m to ΩG. By a suitable choice of boundary conditions we may take m to be CP_1. Now it is the coadjoint orbit that is endowed with the symplectic structure. One can show explicitly that the action associated with the symplectic structure of ΩG is just a

WZW model corresponding to the orbit of the covector $\mathbf{b_0} = (0, \mu)$. In general, there will be an extra term added to the WZW action for those covectors where $\mathbf{b} = (B \neq 0, \mu)$ and which cannot be transported to $\mathbf{b_0} = (0, \mu)$ by loop group action. The existence of these orbits constitutes the difference between ΩG, which is the space of based loop on G, and $\mathcal{G}$, which is the space of free loops on G. Indeed the constant loops which correspond to group elements in G which are not the identity live in $\mathcal{G}$. By choosing the coadjoint orbit with $\mathbf{b_0} = (0, \mu)$ we are looking at the symplectic structure on $\mathcal{G}$ which is invariant under all constant group transformations. Thus $\Omega G = \mathcal{G}/G$ acquires this symplectic structure. Let ω_b be the symplectic two-form associated with the orbit of a covector $\mathbf{b}$. Then the action we seek is simply

$$S = \int_{CP_1} \omega_b.$$

This action for $\mathbf{b_0} = (0, \mu)$ will be defined over maps from $CP_1 \to \Omega G$. In general for a generic covector we may explicitly write,

$$\begin{aligned}
S_0 = & \frac{n}{2\pi} \int Tr \left\{ gBg^{-1} \left[\frac{\partial g}{\partial \lambda} g^{-1}, \frac{\partial g}{\partial \tau} g^{-1} \right] \right\} d\sigma d\tau d\lambda \\
& + \frac{n}{2\pi} \int \mu Tr \left\{ \frac{\partial g}{\partial \sigma} g^{-1} \left[\frac{\partial g}{\partial \lambda} g^{-1}, \frac{\partial g}{\partial \tau} g^{-1} \right] \right\} d\sigma d\tau d\lambda \\
& + \frac{n}{2\pi} \int \mu Tr \left\{ \frac{\partial g}{\partial \lambda} g^{-1} \frac{\partial}{\partial \sigma} \left(\frac{\partial g}{\partial \tau} g^{-1} \right) \right\} d\sigma d\tau d\lambda
\end{aligned}$$

where the integration is over a 3-manifold with $CP_1 \times S^1$ topology. The S^1 factor is coming from the loop integration. Integrating by parts and ignoring total τ derivatives we have

$$
\begin{aligned}
S_0 =& \frac{-n}{2\pi}\int_\Sigma Tr\left(Bg^{-1}\frac{\partial g}{\partial\tau}\right)d\sigma d\tau \\
&-\frac{n\mu}{4\pi}\int_\Sigma Trg^{-1}\frac{\partial g}{\partial\sigma}g^{-1}\frac{\partial g}{\partial\tau}d\sigma d\tau \\
&+\frac{n\mu}{4\pi}\int_{CP_1\times S^1} Trg^{-1}\frac{\partial g}{\partial\sigma}\left[g^{-1}\frac{\partial g}{\partial\lambda},g^{-1}\frac{\partial g}{\partial\tau}\right]d\sigma d\tau d\lambda.
\end{aligned}
$$

Here Σ is a two-dimensional surface that remains after the integration by parts.

Now just as stated the isotropies of the field $\mathbf{b}$ will be used to determine which coadjoint orbit we choose. The isotropy group of $\mathbf{b}$ will arise from those group elements that leave $\mathbf{b}$ invariant. For $\mathbf{b}=(B,\mu)$, this is just the statement that there exists elements, $h(\sigma)\in\mathcal{G}$ such that

$$
h(\sigma):(B(\sigma),\mu)\equiv\left(h(\sigma)Bh^{-1}(\sigma)+\mu\frac{\partial h}{\partial\sigma}h^{-1},\mu\right)=(B,\mu).
$$

Thus the isotropy group of $\mathbf{b}$ specifies the field space of the action. The orbit of $\mathbf{b}$ is thus identified with all those elements of $\mathcal{G}\simeq G\times\Omega G$ modulo H, or in other words $G/H\times\Omega G$. Generally this is because only constant elements in $\mathcal{G}$ will provide isotropy for $\mathbf{b}$ when $\mu\neq 0$. The purely central extension covector, $\mathbf{b_0}=(0,\mu)$, defines the coadjoint orbit that is identified with ΩG.

This established the first phase of our search for an effective QCD action. We have successfully identified a 2D action (in the WZW sense) with that of the instanton sector of Yang-Mills. However, as it stands the action S_0 is not enough for a 4D effective action for QCD. For one thing we need a 4D description and for another we would like to represent the Θ vacua.

We can write an action with these attributes and also incorporate the action S_0 if we consider the immersion of 2D surfaces into a 4D manifold that corresponds to the image of the two surface Σ and the image of the (WZW) manifold with

$CP_1 \times S^1$ topology for the WZ term. One can consider the induced metric and antisymmetric tensors arising from the immersions as the background geometric objects which are coupled to the WZW model with $SU(3)$ Kac-Moody symmetry. Since we are allowing immersions, as opposed to embeddings, self-intersecting surfaces corresponding to the image of Σ can be included. These self-intersecting surfaces will provide the means by which the analogue of the Θ vacua may arise [3].

By writing the WZW model in Lorentz invariant form and expressing the coadjoint vector $\mathbf{b}$ as a world surface vector (recall that the symplectic geometry fixes a gauge), $\mathbf{b} = (B_a, \mu)$, we may write our proposed action as,

$$
\begin{aligned}
I = &\int d^2x Tr B_a g^{-1}\partial_b g(\sqrt{H}H^{ab} + E^{ab}) \\
&+\frac{1}{24\pi}\int d^2x Tr\partial_a g\partial_b g^{-1}\sqrt{H}H^{ab} \\
&+\frac{1}{24\pi}\int d^3y E^{ijk} Tr(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g) \\
&+\Theta\frac{-1}{16\pi}\int d^2x\sqrt{H}H^{ab}\partial_a\tau^{\mu\nu}\partial_b\tau^{\lambda\rho}\epsilon^{\mu\nu\lambda\rho}
\end{aligned}
.
$$

In the above, the induced tensors are defined through the relations,

$$
\begin{aligned}
t_a^\mu &\equiv \partial_a X^\mu(\sigma,\tau) \\
H_{ab} &\equiv t_a^\mu t_b^\nu G_{\mu\nu} \\
E_{ab} &\equiv t_a^\mu t_b^\nu N_A^\lambda N_B^\rho \epsilon^{AB}\epsilon_{\mu\nu\lambda\rho} \\
E_{ijk} &\equiv t_i^\mu(\sigma,\tau,\lambda)t_j^\nu(\sigma,\tau,\lambda)t_k^\lambda(\sigma,\tau,\lambda)\zeta^\rho\epsilon^{\mu\nu\lambda\rho}
\end{aligned}
,
$$

where the latin indices, $a, b \in (1,2)$, $A, B \in (1,2)$ and $i, j, k \in (1,2,3)$, while the Greek indices are 4D space-time indices. The t_a^μ's span the tangent space bundle on the image of Σ, and the N_A^ρ's span the $SO(2)$ normal bundle. For the WZ term we use the tangent vectors, $t_i^\mu, i = 1, 2, 3$ for the image of the $CP_1 \times S^1$ and ζ^ρ as the outward drawn normal to the three surfaces. We denote by H the determinant of H_{ab}. For the sake of generality, we have considered a four manifold with metric $G_{\mu\nu}$ and orientation tensor $\epsilon^{\mu\nu\lambda\rho}$. The first term of the action specifies the space of fields where the action is defined. We are interested in the field space ΩG. Thus we may set $B_a = 0$ and $\mu = \frac{1}{2}$. Setting $\mu = \frac{1}{2}$ and $n = 1$ from the WZW model is sufficient to guarantee a consistent quantum theory [12]. The second and third terms of the action are the usual WZW model coupled to the image of the Riemann surface Σ which is specified by $X^\mu(\sigma, \tau)$. The 3D WZ term is defined on an image of $CP_1 \times S^1$ which is specified by the extended immersion vector $X^\mu(\sigma, \tau, \lambda)$. This WZ term is coupled to the induced antisymmetric rank three tensor, E^{ijk}, on the image of $CP_1 \times S^1$. The last term is the self-intersecting term for the image of the 2D surface, Σ. It is this term that will capture the Θ vacua structure. The self-intersecting surfaces will act as the n-vacua and the instantons will arise as solutions to the Euler-Lagrange equations of the WZW model. For each of the self-intersecting n sectors, there will be the lowest lying state that will serve as the vacuum. From what we know of ordinary strings in 4D, these lowest lying states may correspond to torus knots [13]. We will examine this claim in a later publication.

Let us remark that upon quantization a Liouville mode should emerge. This will break the scale invariance of the theory. We conjecture that this introduction of a scale is necessary and that it should be consistent with the scale for the non-perturbative phase for QCD. Indeed it would be interesting if $\Lambda_{\rm QCD}$ were related to the vacuum expectation value of the Liouville mode from a string theory. Since the scale arises from the anomaly associated with the conformal invariance, after gauge fixing to the light cone gauge the new action will have an extra contribution of [14]

$$I\,'_0 = \frac{1}{21\pi}\int d\sigma d\tau\left[\frac{\partial_\sigma^2 s}{(\partial_\sigma s)^2}\partial_\tau\partial_\sigma s - \frac{(\partial_\sigma^2 s)^2(\partial_\tau s)}{(\partial_\sigma s)^3}\right].$$

In the above $s(\sigma,\tau)$ is the Liouville mode. The coefficient of the above comes from using $n=1, c_v=12$, and Dim $G=8$ in $\frac{2n\ \mathrm{Dim}G}{2n+c_v}$ [14].

We have proposed the action I as a possible effective action for QCD in its non-perturbative phase. The action is motivated by an observation of Atiyah that shows the moduli spaces for instantons and certain 2D maps enjoy a diffeomorphism between them. Physically, we expect the vacuum structure of QCD, with regard to the topological structure of the $SU(3)$ gauge group, to remain unchanged during a phase transition. In other words, the existence of instantons and the Θ vacua are generic properties of the $SU(3)$ bundle over R^4. It is plausible to assume that as QCD undergoes strong coupling into a string-like phase, that the vacuum topological structure remains unaltered. If this is true then one must consider a string theory that, at the very least, preserves the moduli structure of instantons and which has features exhibiting the Θ vacua. We have constructed such a string theory here. The partition function will require integration over the fields $g(\sigma,\lambda,\tau)$ and all possible immersions that admit a differential structure. The partition function will be

$$Z = \int \mathcal{D}X^\mu \mathcal{D}g\ e^{-I(X^\mu, g)}$$

where we have ignored gauge fixing terms and the conformal anomaly. We will address the quantization of this model in a later publication.

As a final remark, one may ask what is the analogue for the $\mathbf{b} = (B \neq 0, \mu)$ orbits in terms of instantons? As we stated earlier, different covectors may correspond to different orbits where the orbits may be identified with $\mathcal{G}/H \simeq G/H \times \Omega G$. Let B be the generator of a homomorphism from the circle S^1 into G. For example when $G = SU(2)$ we may have

$$B = \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix}$$

which generates the homomorphism

$$\alpha(\sigma) = \begin{pmatrix} e^{in\sigma} & 0 \\ 0 & e^{-in\sigma} \end{pmatrix}.$$

Thus B has a $U(1)$ isotropy group. The coadjoint action of $g(\sigma, \lambda, \tau)$ (which includes constant loop transformations which are not the identity) on $\mathbf{b}$ will provide maps from $CP_1 \to G/H \times \Omega G$. Now in a subsequent theorem, Atiyah [4] has shown that there exists an isomorphism between S^1 invariant k instantons which are associated with a homeomorphism α and the parameter space of based holomorphic maps from $CP_1 \to G/H$ of degree k, where H is the centralizer of α. Note that one specifies α in the above theorem since there may be many ways to embed H in G. Thus inequivalent instantons may have the same isotropy group. With this theorem, we are able to identify the orbit of a covector $\mathbf{b}$ with isotropy group H as an axial symmetric instanton. Furthermore, Atiyah has shown that such S^1 invariant k-instantons are monopoles on the 3-space H^3. Our interpretation is then that the other orbits are related to axial symmetric instantons which can be identified with colored monopoles with instanton charge. These monopoles will in general break the global $SU(3)$ color symmetry and this phenomenon is quite reminiscent of the work found in [15] on global color breaking due to monopoles. In general there will be several inequivalent orbits with the same coset space G/H describing the field space due to the number of ways one can embed the subgroup H into

G. Thus we have captured another topological aspect of QCD in this framework, namely magnetic monopoles. Again this is consistent with our philosophy that topological properties associated with the gauge group will have string analogues. For an interesting view of why all instantons are monopoles see Ref. [16].

Just as we expect a Liouville mode in the zero monopole sector we should expect it to appear in the non-zero sector as well. Indeed [14] this is true and one finds that after gauge fixing in the light cone gauge there is an extra contribution to the action due to the conformal anomaly in the monopole sector, i.e. $\mathbf{b}= (B, \frac{1}{2})$ is such that $B \neq 0$ or any pure gauge configuration, that we may write this contribution as

$$
\begin{aligned}
I\,'_B = \frac{1}{21\pi} \int d\sigma d\tau & \left[\frac{\partial_\sigma^2 s}{(\partial_\sigma s)^2} \partial_\tau \partial_\sigma s - \frac{(\partial_\sigma^2 s)^2 (\partial_\tau s)}{(\partial_\sigma s)^3} \right] \\
& + \int B(\sigma) \left(\frac{\partial_\lambda s}{\partial_\sigma s} \partial_\sigma \left[g^{-1} \partial_\tau g \right] - \frac{\partial_\tau s}{\partial_\sigma s} \partial_\sigma \left[g^{-1} \partial_\lambda g \right] \right) d\sigma \; d\lambda \; d\tau \\
& + \int B(\sigma) \left[g^{-1} \partial_\lambda g, g^{-1} \partial_\tau g \right] d\sigma \; d\lambda \; d\tau .
\end{aligned}
$$

In the above $g = g(\sigma, \lambda, \tau)$. The last two terms resemble a BF system where the derivative operators are modified by the gauge fixed 3D extended metric so that $g^{-1}\partial g$ is no longer pure gauge. We will try to clarify the role of these monopoles as BF and Chern-Simons theories in Ref. [14].

The work above can of course be put on a more general footing than QCD. One can study the instanton sector of other theories using these ideas. Finally we would like to note that P.A. Griffin has also considered the role of WZW models in explaining the non-perturbative phase of QCD through lattices techniques [17].

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